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ROBUST DESIGN AND ROBUST STABILITY ANALYSIS OF ACTIVE NOISE CONTROL SYSTEMS

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The problem of designing robust active control systems is addressed in this paper. A variety of active control design problems are formulated as semidefinite programming (SDP) problems. An SDP problem is a convex optimization problem, consisting of a linear objective function subject to linear matrix inequality (LMI) constraints. First, an SDP formulation is presented for the design of multichannel LMS algorithms with limited-capacity secondary sources. Simulations show that this SDP formulation is an order of magnitude more computationally efficient than the usual non-linear constrained optimization formulations. Secondly, the design of robust LMS algorithms is presented as an SDP problem. These algorithms minimize the worst-case control error in the presence of unknown but norm-bounded perturbations on the secondary path model and on the primary field. Both the unstructured and structured perturbations cases are considered. The resulting controllers are exact solutions to the robust control design problem, except in the most general case of structured perturbations when they only minimize an upper bound on the worst-case residual control error. Thirdly, SDP formulations are proposed to compute guaranteed stability limits for the adaptive multiple-channel leaky LMS algorithm in the presence of both unstructured and structured perturbations on the secondary path. Monte Carlo simulations show that the obtained stability limits are much more reliable than previously used limits, based, for example, on the Gershgorin circle theorem.

1. INTRODUCTION

The adaptive feedforward LMS algorithm, as first proposed by Elliott *et al.* [1], is widely used in multiple channel active noise control systems. Such large-scale active noise control systems are now commonly installed, for example, in the passenger cabin of propeller aircraft [2], where the disturbing sound field is predominantly tonal. The LMS algorithm drives a number of secondary sources, usually loudspeakers or electrodynamic shakers which are appropriately distributed over the cabin or the aircraft fuselage, so as to minimize the sum of the squared signals from the error microphones, located in the area to be silenced. The algorithm relies on the knowledge of the transfer function matrix from the secondary sources to the error microphones (the so-called secondary path transfer function matrix) to generate the optimal control signals. However, this secondary path transfer function matrix is subject to changes under operational circumstances due to, for example, changing flight conditions, fuselage pressurization, passengers moving around in the cabin, etc. One way to cope with that problem is to measure this transfer function on a regular basis during operation of the control system, and adapt the controller accordingly. The

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drawback of this on-line identification approach (as proposed by Bao [3]) is, first, the need for injecting additional noise in the system, and, second, the significantly increased computational load on the control system signal processing unit. An alternative way is to design a control algorithm which is robust to such changes.

Robust control systems are generally required to exhibit two types of robustness properties [4]. *Robust stability* means that the control system remains stable under changing operational conditions, while *robust performance* means that the control system achieves its required performance under changing conditions.

Boucher *et al.* [5], Elliott *et al.* [6], and Elliott [7] discuss the robust stability analysis of the adaptive LMS algorithm for perturbations on the secondary path model. They only derive guaranteed stability limits (in terms of classes of perturbations under which the control system is guaranteed to remain stable) for the single-channel case. In the multiple-channel case, they conduct Monte Carlo simulations for different relative perturbation levels of the secondary path, and investigate the probability of instability, without proposing guaranteed theoretical stability limits. They show that the robust stability of the LMS algorithm can be improved by applying some effort weighting (or leak). Omoto and Elliott [8] further study the robust stability of multichannel feedforward controllers when the secondary path is affected by structured perturbations. They also characterize how different types of physical perturbations of the secondary path produce structured perturbations of the singular value matrices of the secondary path. They propose two stability criterions, one appearing to be too optimistic and the other one, which is based on the Gershgorin circle theorem [9], to be over conservative. This paper makes a significant contribution to this research area by presenting sufficient conditions for the robust stability of the adaptive multiple-channel LMS algorithm, in the case of both unstructured and structured perturbations of the secondary path transfer function matrix. Simulations show that the obtained stability limits are much more reliable than those given by Omoto and Elliott [8]. The proposed methods also allow the smallest leak parameter to be chosen which still guarantees the stability of the adaptive algorithm in the case of a bounded, structured set of perturbed secondary paths. As such, they solve the robust stability *analysis* and *design* problem for the adaptive leaky LMS algorithm.

The robust performance issue for the single-tone multiple channel LMS algorithm is also addressed in this paper. Research in this area has mainly focussed on the effect of secondary path perturbations on the performance of LMS algorithms, and on the beneficial influence of leak (see e.g. references [5, 6, 8]). Here, the robust performance *design* problem, instead of the robust performance *analysis* problem, is treated. An algorithm that minimizes the worst-case residual control error under additive perturbations, which are unknown but bounded in maximum singular value norm, is presented. The perturbations are either unstructured or structured, in which case they affect the secondary path closely. The proposed method yields an exact solution in the case of a full unstructured uncertainty block, but only a close upper bound to the exact solution in the case of structured uncertainty.

The robust performance and the robust stability analysis problem are both formulated as convex semidefinite programming (SDP) problems. SDP problems are convex optimization problems, and consist in minimizing a real linear objective function of a vector of variables $\mathbf{x} \in \Re^m$, subject to linear real-valued matrix inequality (LMI) constraints [10]:

$$\min \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

$$\text{s.t.} \quad \mathbf{F}(\mathbf{x}) \geqslant 0, \tag{1}$$

where

$$\mathbf{F}(\mathbf{x}) \equiv \mathbf{F}_0 + \sum_{i=1}^{m} \mathbf{x}_i \mathbf{F}_i \tag{2}$$

The problem data are the vector $\mathbf{c} \in \Re^m$ and $m + 1$ symmetric matrices $\mathbf{F}_0, \ldots, \mathbf{F}_m \in \Re^{n \times n}$. The inequality sign '$\geqslant 0$' in equation (1) means that $\mathbf{F}(\mathbf{x})$ is positive semidefinite. Similarly, the inequality sign '> 0' following a matrix means that this matrix is positive definite.

Such SDP problems are easily and efficiently solved in polynomial time using the recently developed interior point methods [11, 12]. All simulations in this paper have been carried out using the software package SDPT3 [12], a Matlab implementation of infeasible path-following algorithms for solving standard SDP problems. This code does not exploit any particular problem structure, except for sparsity and block-diagonal structure of data matrices.

The next section introduces the system under study, and the uncertainty models which are used to describe the possible perturbations on the nominal system. In section 3, the deterministic multiple-channel LMS control design problem is formulated as an SDP problem, taking into account the possible limited capacity of the control sources. Section 4 defines the robust LMS control problem and presents its solution as an SDP problem, first for the unstructured, and later also for the structured perturbations case. Section 5 deals with the robust stability analysis of the adaptive multiple channel LMS algorithm. Finally, some conclusions are drawn in the last section.

The paper presents a theoretical framework which is the result of a synthesis of concepts from different scientific fields such as active noise and vibration control, robust control and convex optimization. Illustrative simulations support and clarify the theoretical contributions in the paper.

2. DESCRIPTION OF THE SIMULATION AND PERTURBATION MODELS

2.1. DEFINITION OF THE SIMULATION MODEL

The system under study in this paper is a box-like acoustical volume, representing a room, with rigid walls except for a flexible glass plate, incorporated in one of the side walls and representing a window. This set-up, which was originally designed for another study, is shown in Figure 1. The dynamic behaviour of this system is analyzed by a coupled vibro-acoustic finite element model. The acoustical subsystem is modelled with 42 eight-noded solid elements in the x direction, 18 in the y direction and 28 the in z direction, resulting in 21 168 elements in total. The room is filled with air with a speed of sound of 340 m/s and a density of 1·225 kg/m^3. The glass plate is modelled by 160 four-noded plate elements, 16 in the x direction and 10 in the y direction. The density of the glass is 2500 kg/m^3, the Young's modulus 72 GPa and the Poisson coefficient 0·21. Transfer functions are calculated using the modal superposition technique with a set of 43 coupled eigenmodes with eigenfrequencies up to 150 Hz, assuming a modal damping of 2% for all modes. A sound field in the room is generated by a set of 20 randomly incident plane waves on the window at 87 Hz.

An active noise control system is installed in this room to suppress the primary field. The baseline system drives eight control sources, located in the eight corners of the room, so as to reduce the acoustic field measured by nine error microphones in a horizontal plane at 1·5 m above the ground. The locations of these sensors and control sources are also indicated in Figure 1. If control systems with different numbers of sensors or control sources are used in the simulations throughout the paper, the control sources are randomly chosen

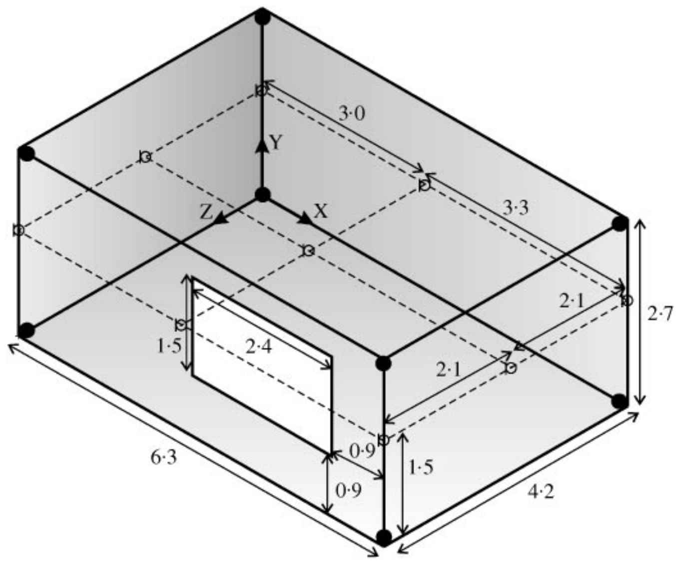

Figure 1. Schematic view of the system under study. All dimensions are given in meters. ●, control source; ○, error microphone.

on the six walls of the room, and the sensors are randomly chosen in the horizontal plane at 1·5 m above the ground.

2.2. DESCRIPTION OF THE UNCERTAINTY MODELS

This section introduces the basic models for representing unknown or unmodelled variations of a system around its known nominal behaviour. The term "uncertainty" generally refers to unknown or unpredictable model variations, while the term "perturbation" refers to known or predictable model variations, such as non-linear effects which cannot be dealt with in a linear control theory. In the following section a clear distinction is no longer made between both terms as they are treated in exactly the same way. The basic concept in modelling an uncertain system is to separate what is known from what is unknown in a feedback-like connection, known as a linear-fractional representation, and bound the possible values of the unknown elements [4]. Such a linear-fractional representation allows perturbed transfer function matrices to be modelled whose elements depend rationally on the independent perturbation sources. In this paper, only close dependencies of the perturbed secondary path on the perturbations are considered. The perturbed transfer function matrix $\mathbf{G}$ is written as

$$\mathbf{G} = \mathbf{G}_0 + \mathbf{\Delta G}, \tag{3}$$

where $\mathbf{\Delta G}$ is an additive perturbation on the nominal transfer function matrix $\mathbf{G}_0$.

In the simple unstructured perturbations case, the perturbed transfer function matrix $\mathbf{G}$ belongs to a sphere with radius ρ around $\mathbf{G}_0$:

$$\mathbf{G} = \mathbf{G}_0 + \rho\mathbf{\Delta}, \tag{4}$$

where the perturbation block is only known to be unstructured and bounded in norm $\|\mathbf{\Delta}\| \leqslant 1$.

Throughout this paper, the maximum singular value norm is used for matrix norms, and the Euclidean norm for vectors. The parameter ρ is the perturbation size. In case the primary field $\mathbf{d}$ is also affected by perturbations, the unstructured uncertainty model takes the form

$$[\mathbf{G}\ \mathbf{d}] = [\mathbf{G}_0\ \mathbf{d}_0] + \rho\mathbf{\Delta} \quad \text{with} \quad \mathbf{\Delta} = [\mathbf{\Delta G}\ \ \mathbf{\Delta d}] \quad \text{and} \quad \|\mathbf{\Delta}\| \leqslant 1. \tag{5}$$

Before proceeding to the structured perturbations case, consider first the singular value decomposition of the nominal secondary path

$$\mathbf{G}_0 = \mathbf{U}\mathbf{\Sigma}_0\mathbf{V}^H. \tag{6}$$

Using the left and right singular vectors of the nominal secondary path $\mathbf{G}_0$, it is possible to express the perturbation as

$$\mathbf{\Delta G} = \mathbf{U}\mathbf{\Delta\Sigma}\mathbf{V}^H, \tag{7}$$

where $\mathbf{\Delta\Sigma}$ is not necessarily diagonal.

Omoto and Elliott [8], and Baek and Elliott [13] investigated how $\mathbf{\Delta\Sigma}$ is affected by different types of physical perturbations on a system similar to the one depicted in Figure 1. Baek and Elliott [13] observed that the effect of structured perturbations, such as diffracting objects in the acoustic volume, mainly appears in the upper left part of $\mathbf{\Delta\Sigma}$. Therefore, in the present study it is also assumed that the magnitude of each element in $\mathbf{\Delta\Sigma}$ is inversely proportional to its distance from the top left element, in agreement with conclusions drawn by Baek and Elliott [13]. This structure in the maximum magnitude of each element of $\mathbf{\Delta\Sigma}$ is modelled in the (normalized) weighting matrix $\mathbf{W}$:

$$\hat{\mathbf{W}}_{i,j} = \sqrt{\frac{2}{i^2 + j^2}} \quad \text{for } i = 1, \dots, N_e \quad \text{and} \quad j = 1, \dots, N_c \quad \text{and} \quad \mathbf{W} = \frac{\hat{\mathbf{W}}}{\|\hat{\mathbf{W}}\|}. \tag{8}$$

where N_e, and N_c, are respectively the number of error sensors and control sources in the system. According to Baek and Elliott [13], the phase of each element of $\mathbf{\Delta\Sigma}$ is an independent random perturbation, which results in a total of $N_e \times N_c$ random perturbations. This number is too high from a practical point of view, and therefore, it is assumed here that the independent perturbations affect only the secondary path via the N_e sensor outputs. This latter assumption then yields the following structured perturbations model for the singular-value matrix:

$$\mathbf{\Delta\Sigma} = \rho\mathbf{\Delta}\mathbf{W}_m,$$

and

$$\mathbf{\Delta} = \mathrm{diag}(\delta_1, \dots, \delta_{N_e}), \text{ with } |\delta_i| \leqslant 1 \quad \text{for } i = 1, \dots, N_e, \tag{9}$$

where the perturbation size ρ controls the magnitude of the largest element in $\mathbf{\Delta\Sigma}$. In case the primary field is also subject to uncertainty, the structured perturbations model is extended with

$$\mathbf{\Delta d} = \mathbf{U}\mathbf{\Delta\Sigma}\mathbf{V}_d^H, \tag{10}$$

where $\mathbf{V}_d$ is arbitrarily constructed by averaging the rows of $\mathbf{V}$ and multiplying the resulting vector by $\|\mathbf{d}\|$. The underlying assumption is that the independent perturbations affect the response of the system on the primary excitation in the same way as that on the secondary excitation.

Note that this choice of structure in the perturbations is quite arbitrary, but it only serves to illustrate the theoretical contributions of the paper. In practice, structure in the perturbations depends on the actual application being considered. The first step in designing a robust controller is always the identification of an appropriate nominal model *and* of a representative model of the uncertainty or perturbations on that nominal model.

3. SDP FORMULATION OF THE QUADRATIC CONTROLLER MODEL

This section defines the deterministic LMS controller design problem with effort constraints on the individual control sources, as treated by Elliott and Baek [14], as an SDP problem. The vector of signals $\mathbf{e}$ measured by the error sensors (the residual field) consists of a component due to the disturbing sound field (the primary field), $\mathbf{d}$, and a component due to the control action (the secondary field), $\mathbf{Gu}$, where $\mathbf{G}$ is the secondary path transfer function matrix and $\mathbf{u}$ is the vector of control signals, such that

$$\mathbf{e} = \mathbf{d} + \mathbf{Gu}. \tag{11}$$

In the case of a tonal excitation, the multiple-channel LMS controller yields the solution to the following quadratic optimization problem [6]:

$$\begin{aligned} &\min_{\sigma, u} \quad \sigma \\ &\text{s.t.} \quad \sigma = (\mathbf{d} + \mathbf{Gu})^H(\mathbf{d} + \mathbf{Gu}) \end{aligned} \tag{12}$$

with $\mathbf{d}$, $\mathbf{G}$ and $\mathbf{u}$ being complex-valued vectors or matrices and σ a real scalar. The complex optimization problem (12) is first converted into one involving only real-valued matrices and vectors. Therefore, it is convenient to introduce the real counterparts of the complex $\mathbf{e}$, $\mathbf{d}$, $\mathbf{u}$, and $\mathbf{G}$ matrices

$$\mathsf{e} = \begin{Bmatrix} \mathrm{Re}(\mathbf{e}) \\ \mathrm{Im}(\mathbf{e}) \end{Bmatrix}, \quad \mathsf{d} = \begin{Bmatrix} \mathrm{Re}(\mathbf{d}) \\ \mathrm{Im}(\mathbf{d}) \end{Bmatrix}, \quad \mathsf{u} = \begin{Bmatrix} \mathrm{Re}(\mathbf{u}) \\ \mathrm{Im}(\mathbf{u}) \end{Bmatrix}, \quad \text{and} \quad \mathsf{G} = \begin{bmatrix} \mathrm{Re}(\mathbf{G}) & -\mathrm{Im}(\mathbf{G}) \\ \mathrm{Im}(\mathbf{G}) & \mathrm{Re}(\mathbf{G}) \end{bmatrix}. \tag{13}$$

The real-valued counterpart of equation (11) then becomes

$$\mathsf{e} = \mathsf{Gu} + \mathsf{d}. \tag{14}$$

After some simple algebraic manipulations and the use of Schur complements (see Appendix A), the complex quadratic optimization problem, equation (12), can be posed as the following SDP problem in σ and $\mathbf{u}$ with one LMI constraint:

$$\begin{aligned} &\min_{\sigma, u} \quad \sigma \\ &\text{s.t.} \quad \begin{bmatrix} \mathsf{I} & (\mathsf{d} + \mathsf{Gu}) \\ (\mathsf{d} + \mathsf{Gu})^{\mathrm{T}} & \sigma \end{bmatrix} \geqslant 0. \end{aligned} \tag{15}$$

Possible limits on the individual control signals to the secondary sources, as introduced by Elliott and Baek [14], are easily incorporated in this SDP formulation after conversion into additional LMI constraints. Consider the limitation of the uth complex control signal

$$|\mathbf{u}_u| \leqslant |\mathbf{u}_u^{max}|. \tag{16}$$

The real and imaginary parts of this control signal are selected from the real vector $\mathbf{u}$ by means of a selection matrix $\boldsymbol{\Sigma}_u$

$$\begin{bmatrix} \mathbf{u}_u \\ \mathbf{u}_{u+N_u} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \end{bmatrix} \mathbf{u} = \boldsymbol{\Sigma}_u \mathbf{u}. \tag{17}$$

Using equation (17), inequality (16) turns into the following inequality:

$$(\boldsymbol{\Sigma}_u \mathbf{u})^{\mathrm{T}} (\boldsymbol{\Sigma}_u \mathbf{u}) \leqslant |\mathbf{u}_u^{max}|^2. \tag{18}$$

The application of the Schur complements of Appendix A allows the conversion of the non-linear inequality (18) into the LMI constraint

$$\begin{bmatrix} \mathbf{I}|\mathbf{u}_u^{max}| & (\boldsymbol{\Sigma}_u \mathbf{u}) \\ (\boldsymbol{\Sigma}_u \mathbf{u})^{\mathrm{T}} & |\mathbf{u}_u^{max}| \end{bmatrix} \geqslant 0. \tag{19}$$

After the conversion of the real vector $\mathbf{u}$, which is the solution to the problem (15) subject to an appropriate number of constraints (19), into the complex vector $\mathbf{u}$ with the bounded complex control signals, the Lagrange multipliers of Elliott and Baek (equation 5 in reference [14]) in the diagonal matrix $\boldsymbol{\Lambda}$ are recovered by solving the complex set of uncoupled algebraic equations

$$\boldsymbol{\Lambda}\mathbf{u} = -\mathbf{G}^H(\mathbf{d} + \mathbf{G}\mathbf{u}). \tag{20}$$

The resulting controller can then be implemented by the following adaptive algorithm

$$\mathbf{u}(k+1) = (I - \alpha\boldsymbol{\Lambda})\mathbf{u}(k) - \alpha\mathbf{G}^H\mathbf{e}(k), \tag{21}$$

where α is the convergence coefficient.

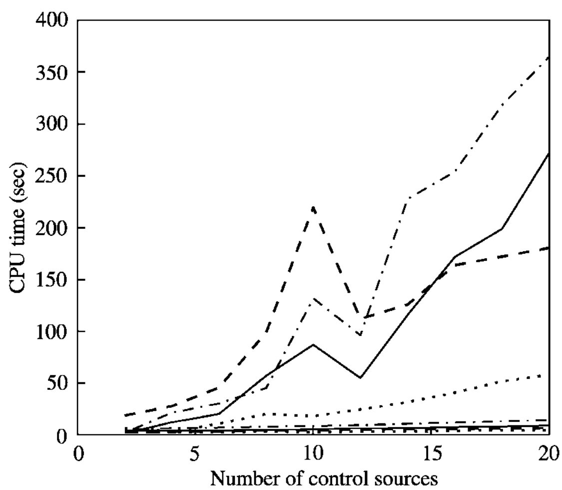

Figure 2. Computation times for solving the optimal LMS control problem with different numbers of limited-capacity control sources and error sensors (·········, 5; – – – –, 10; ——, 20; – · – · – · –, 40), in bold lines with SDP methods and in grey lines with classical non-linear optimization methods.

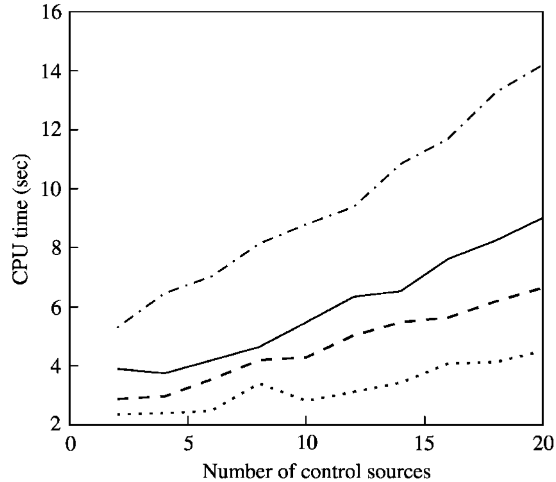

Figure 3. Computation times for solving the optimal LMS control problem with different numbers of limited-capacity control sources and error sensors (········, 5; – – – –, 10; ——, 20; –·–·–·–, 40) using SDP methods.

Theoretically, there is no difference between the Kuhn–Tucker formulation of this problem (in reference [14]) and the SDP formulation of equations (15) and (19). The Lagrange multipliers on the diagonal of the (diagonal) matrix $\mathbf{\Lambda}$ indeed appeared to be real numbers in the simulations. The main advantage of the SDP formulation of the optimal control problem with effort constraints on the individual control signals is its superior computational efficiency in comparison with classical iterative methods. Figure 2 compares the computation times for solving this constrained optimal control problem with a classical sequential quadratic programming method (SQP, see reference [15]) in grey lines and with the SDP method in bold lines, for control systems with different numbers of control sources and error sensors. The SDP method is more than ten times faster than the classical method for solving problems with a large number of control sources and error sensors, occurring for example in active aircraft cabin noise reduction. Figure 3 shows the same computation times, but for the SDP method only. The required computation times increase only linearly with the number of control sources in the system.

4. SYNTHESIS OF ROBUST LMS CONTROLLERS

4.1. THE UNSTRUCTURED PERTURBATIONS CASE

The previous section showed how the nominal deterministic multiple-channel LMS algorithm with limited-capacity secondary actuators can be formulated as an SDP problem. Here, it is assumed that both the secondary path transfer function matrix $\mathbf{G}$ and the primary field vector $\mathbf{d}$ are subject to unknown but bounded and (not necessarily small) deterministic perturbations, and it will be shown how the problem of finding a controller which minimizes the worst-case residual, can also be posed as an SDP problem. The results presented here are essentially based on the application of the theoretical results by El Ghaoui and Lebret [16].

Consider the perturbed system defined in equation (5). The robust performance analysis problem amounts to computing the worst-case residual control error for a fixed control signal vector:

$$\sigma(\mathbf{G}_0, \mathbf{d}_0, \rho, \mathbf{u}) = \max_{\|\Delta\| \leqslant \rho} \|(\mathbf{G}_0 + \Delta\mathbf{G})\mathbf{u} + (\mathbf{d}_0 + \Delta\mathbf{d})\|. \tag{22}$$

The optimal robust controller minimizes the function σ in equation (22), thereby trading off accuracy for robustness (with respect to the assumed perturbation model):

$$\sigma_{opt}(\mathbf{G}_0, \mathbf{d}_0, \rho) = \min_{u} \max_{\|\Delta\| \leqslant \rho} \|(\mathbf{G}_0 + \Delta\mathbf{G})\mathbf{u} + (\mathbf{d}_0 + \Delta\mathbf{d})\|. \tag{23}$$

According to El Ghaoui and Lebret [16], the minimax problem (23) can then be posed as a convex second order cone programming (SOCP) problem. Using Schur complements, this SOCP formulation is easily converted into an SDP formulation for solving problem (23).

$$\begin{aligned} &\min_{\sigma,\tau,u} \sigma \\ &\text{s.t.} \quad \begin{bmatrix} (\sigma - \tau)\mathbf{I} & (\mathbf{G}_0\mathbf{u} + \mathbf{d}_0) \\ (\mathbf{G}_0\mathbf{u} + \mathbf{d}_0)^{\mathrm{T}} & (\sigma - \tau) \end{bmatrix} \geqslant 0 \\ &\qquad \begin{bmatrix} \tau\mathbf{I} & \rho\begin{bmatrix}\mathbf{u} \\ \mathbf{1}\end{bmatrix} \\ \rho\,[\mathbf{u}^{\mathrm{T}} \;\; \mathbf{1}] & \tau \end{bmatrix} \geqslant 0. \end{aligned} \tag{24}$$

At the optimum, the variable τ takes the following value:

$$\tau_{opt} = \rho\sqrt{\|\mathbf{u}\|^2 + 1}. \tag{25}$$

The solution to the original SOCP problem and to the SDP problem (24) is given by

$$\mathbf{u} = \begin{cases} -(\mu\mathbf{I} + \mathbf{G}_0^{\mathrm{T}}\mathbf{G}_0)^{-1}\mathbf{G}_0^{\mathrm{T}}\mathbf{d}_0 & \text{if } \mu \equiv \rho^2\dfrac{(\sigma - \tau)}{\tau} > 0, \\ -\mathbf{G}_0^{+}\mathbf{d}_0 & \text{else } (\mu = 0). \end{cases} \tag{26}$$

The degenerate case ($\mu = 0$) occurs when $\mathbf{d}_0$ is in the column space of $\mathbf{G}_0$, and the corresponding pseudo-inverse solution yields the minimum-norm control signal vector. In the non-degenerate case, the solution has the same form as that given by Elliott *et al.* (equation 7 in reference [6]), where the leak parameter β was introduced to improve the robustness of the controller and was chosen *a priori*. Here, the parameter μ plays the same role, but it is the *a posteriori* result of the robust control design problem (24). This formulation allows an optimal choice to be made, from a robustness point of view, for the leak parameter μ with respect to the assumed perturbation size. Figure 4 shows the optimal μ, from a robust performance point of view, as a function of the assumed perturbation size ρ for the control system with eight control sources and nine error sensors. The horizontal dashed lines represent the singular values of the secondary path transfer matrix $\mathbf{G}_0$. This result is validated by means of a Monte Carlo simulation of 1000 perturbed systems with a perturbation size ρ equal to 10^{-5}. Figure 5 shows the distribution of the primary field attenuations achieved by the robust controller (26) with $\mu \cong 10^{-9}$ in light bars, and by the nominal controller in dark bars ($\mu = 0$). The robust controller achieves a much higher average attenuation, and the variance on the attenuations is also much smaller.

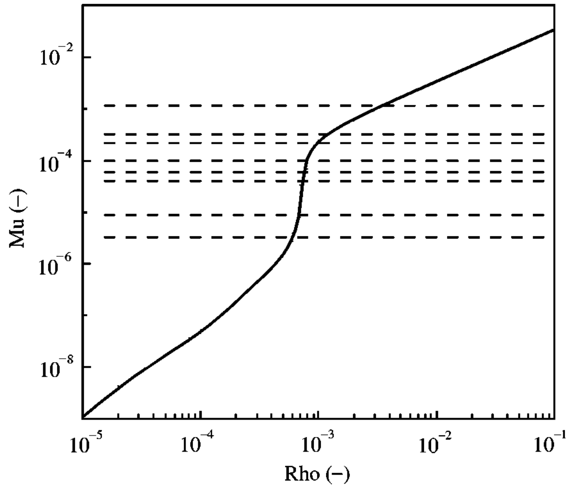

Figure 4. Evolution of the optimal leak parameter for robust performance in the face of unstructured perturbations of secondary path and primary field, as a function of perturbation size. The dashed grey lines represent the singular values of the secondary path.

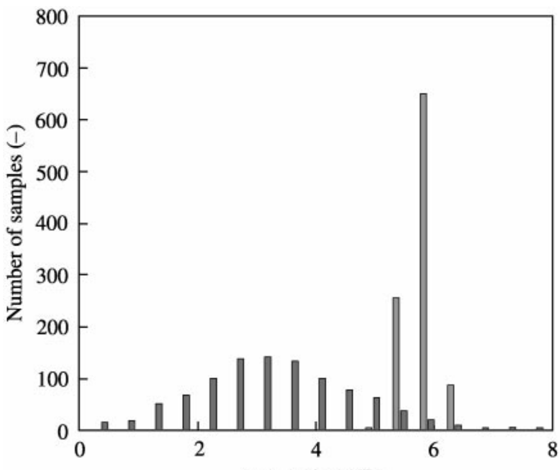

Figure 5. Distribution of the primary field attenuations achieved by the robust controller with $\mu \cong 10^{-9}$ in light bars, and by the nominal controller in dark bars ($\mu = 0$), over 1000 perturbed systems (perturbon size $\rho = 10^{-5}$).

In case the primary field is not affected by the perturbations ($\mathbf{\Delta d} = \mathbf{0}$ in equation (5)), the robust controller is the solution to the SDP problem (24), with the second LMI condition replaced by

$$\begin{bmatrix} \tau \mathbf{I} & \rho \mathbf{u} \\ \rho \mathbf{u}^{\mathrm{T}} & \tau \end{bmatrix} \geqslant 0. \tag{27}$$

An additional advantage of the SDP formulation (24) is the fact that other constraints, such as the limitations on the individual control signals in equation (19), can easily be incorporated in the problem.

4.2. THE STRUCTURED PERTURBATIONS CASE

In this section, it is assumed that the perturbations on both the secondary path $\mathbf{G}$ and the primary field $\mathbf{d}$ depend closely on the unknown uncertainty matrix $\boldsymbol{\Delta}$:

$$[\mathbf{G}(\boldsymbol{\Delta})\ \mathbf{d}(\boldsymbol{\Delta})] = [\mathbf{G}_0\ \mathbf{d}_0] + \rho\mathbf{L}\boldsymbol{\Delta}[\mathbf{R}_G\ \mathbf{R}_d]. \tag{28}$$

The uncertainty matrix is bounded in norm (by unity), and structured. The parameter ρ again determines the size of the perturbations. Let $\boldsymbol{\Delta}_S$ be a subspace of $\Re^{N\times N}$ with the same structure as $\boldsymbol{\Delta}$, and where N equals the number of independent uncertainty sources in the system. Define the worst-case residual as in equation (19):

$$\sigma(\mathbf{G}_0, \mathbf{d}_0, \mathbf{u}) = \max_{\substack{\boldsymbol{\Delta}\in\boldsymbol{\Delta}_S \\ \|\boldsymbol{\Delta}\|\leqslant 1}} \|\mathbf{G}(\boldsymbol{\Delta})\mathbf{u} + \mathbf{d}(\boldsymbol{\Delta})\|. \tag{29}$$

Computing this worst-case residual is difficult, in the sense that it cannot be performed in polynomial time (the problem is NP-hard). Just as for the robust performance analysis of feedback systems, it is possible to compute an upper bound on the worst-case residual in polynomial time, and the concept of scaling the uncertainty block is also applied here in order to minimize this upper bound using SDP. This theoretical result establishes a parallel between the μ-analysis and synthesis procedure for designing robustly performing feedback systems [4], and the SDP formulation for designing (single-tone) multi-channel LMS algorithms under a robust performance requirement.

The minimization of the upper bound on equation (29) requires the introduction of some additional linear subspaces (for scaling purposes):

$$\begin{aligned} \mathsf{B} &\equiv \{\mathbf{B}\in\Re^{N\times N} | \mathbf{B}\boldsymbol{\Delta} = \boldsymbol{\Delta}\mathbf{B}, \forall\boldsymbol{\Delta}\in\boldsymbol{\Delta}_S\}, \\ \mathsf{S} &\equiv \{\mathbf{S}\in\mathsf{B} | \mathbf{S} = \mathbf{S}^{\mathrm{T}}\}, \\ \mathsf{T} &\equiv \{\mathbf{T}\in\mathsf{B} | \mathbf{T} = -\mathbf{T}^{\mathrm{T}}\}. \end{aligned} \tag{30}$$

El Ghaoui and Lebret [16] formally prove that a minimal upper bound λ on the optimal worst-case residual can be obtained by solving the following SDP problem:

$$\min_{\mathbf{S},\mathbf{T},\lambda,u} \ \lambda$$

$$\text{s.t.} \quad \mathbf{S}\in\mathsf{S}, \mathbf{T}\in\mathsf{T},$$

and

$$\left[\begin{array}{c|c} \boldsymbol{\Theta} & \begin{matrix}(\mathsf{G}_0\mathbf{u} + \mathsf{d}_0) \\ (\mathsf{R}_G\mathbf{u} + \mathsf{R}_d)\end{matrix} \\ \hline (\mathsf{G}_0\mathbf{u} + \mathsf{d}_0)^{\mathrm{T}}\ (\mathsf{R}_G\mathbf{u} + \mathsf{R}_d)^{\mathrm{T}} & \lambda \end{array}\right] > 0, \tag{31}$$

with

$$\boldsymbol{\Theta} = \begin{bmatrix} \lambda\mathbf{I} - \rho^2\mathsf{L}\mathbf{S}\mathsf{L}^{\mathrm{T}} & \rho\mathsf{L}\mathbf{T} \\ \rho\mathbf{T}^{\mathrm{T}}\mathsf{L}^{\mathrm{T}} & \mathbf{S} \end{bmatrix}$$

where L, R_G, and R_d are the real-valued equivalents of $\mathbf{L}$, $\mathbf{R}_G$, and $\mathbf{R}_d$, constructed in the same way as in equation (13). The matrices $\mathbf{S}$ and $\mathbf{T}$ are scaling matrices which aim to reduce the effect of the perturbations on the outcome of the SDP problem as much as possible. They play a similar role as the D-scaling matrices in the computation of the structured singular value (or μ) in the robust stability analysis of state space models subject to structured perturbations via a linear-fractional transformation [4].

If $\mathbf{\Theta} > 0$ at the optimum, the upper bound is exact. Due to the strictly positive definiteness of $\mathbf{\Theta}$, its inverse exists, and if $\mathbf{R}_G$ is of full rank, the solution to the SDP problem (31) can be interpreted as the weighted least-squares solution for the following augmented problem:

$$\min_u Z = [(\mathbf{G}_0\mathbf{u} + \mathbf{d}_0)^{\mathrm{T}} \ (\mathbf{R}_G\mathbf{u} + \mathbf{R}_d)^{\mathrm{T}}]\mathbf{\Theta}^{-1}\begin{bmatrix}(\mathbf{G}_0\mathbf{u} + \mathbf{d}_0)\\(\mathbf{R}_G\mathbf{u} + \mathbf{R}_d)\end{bmatrix}. \tag{32}$$

In that case, the design of a multi-channel LMS algorithm for robust performance consists in solving the SDP problem (32) for the weighting matrix $\mathbf{\Theta}$ and the weighted augmented quadratic optimization problem (31).

Consider now the case when the uncertainty block $\mathbf{\Delta}$ is diagonal, and the matrices $\mathbf{L}$, $\mathbf{R}_G$ and $\mathbf{R}_d$ are obtained from the structured perturbation model of section 2.2:

$$\mathbf{L} = \mathbf{U}, \quad \mathbf{R}_G = \mathbf{W}\mathbf{V}^H \quad \text{and} \quad \mathbf{R}_d = \mathbf{W}\mathbf{V}_d^H. \tag{33}$$

As a result of the uncertainty block being diagonal, the matrix $\mathbf{S}$ is diagonal too, the matrix $\mathbf{T}$ equals the null matrix, and the matrix $\mathbf{\Theta}$ takes the following form:

$$\mathbf{\Theta} = \begin{bmatrix}\lambda\mathbf{I} - \rho^2\mathbf{S} & \mathbf{0}\\ \mathbf{0} & \mathbf{S}\end{bmatrix}. \tag{34}$$

This result is validated by means of a Monte Carlo simulation of 1000 perturbed systems with a perturbation size ρ equal to 2×10^{-4}. Figure 6 compares the distributions of the primary field attenuations achieved by the unstructured perturbations robust controller in dark bars, and by the structured perturbations robust controller in light bars. Obviously, ignoring structure in the perturbations leads to a controller which is robust against a much wider variety of possible system perturbations than those actually occurring. Such a controller achieves only smaller reductions than a controller which is specifically designed for robustness against perturbations with a particular structure.

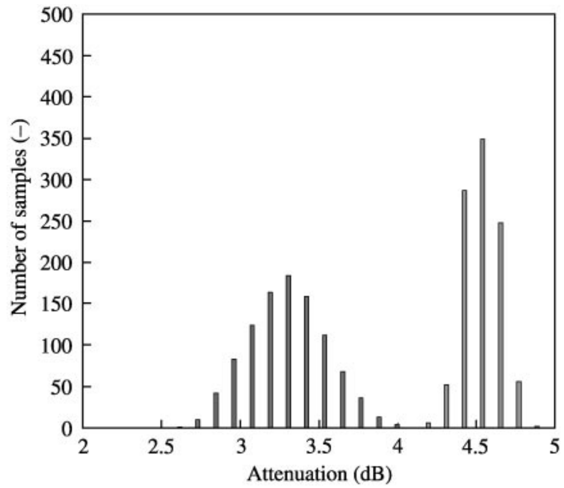

Figure 6. Distributions of the primary field attenuations achieved by the unstructured perturbations robust controller in dark bars, and by the structured perturbations robust controller in light bars, over 1000 perturbed systems with a perturbation size $\rho = 2 \times 10^{-4}$.

In case the primary field is not affected by the perturbations ($\mathbf{R}_d = \mathbf{0}$), the quadratic optimization problem (32) can be written as

$$\min_u \sigma = \mathsf{e}^{\mathrm{T}}\mathsf{Q}\,\mathsf{e} + \mathsf{u}^{\mathrm{T}}\mathsf{R}\mathsf{u}. \tag{35}$$

The weighting matrices, $\mathbf{Q}$ and $\mathbf{R}$, are optimized for the given robustness model by solving the SDP problem (31) for $\mathbf{S}$, and take the following form:

$$\mathbf{Q} = (\lambda\mathbf{I} - \rho^2\mathbf{S})^{-1} \tag{36}$$

and

$$\mathbf{R} = \mathsf{V}\mathsf{W}^{\mathrm{T}}\mathsf{S}^{-1}\mathsf{W}\mathsf{V}^{\mathrm{T}}, \tag{37}$$

where V and W are the real-valued equivalents of $\mathbf{V}$ and $\mathbf{W}$, constructed in the same way as in equation (13). The controller objective (35) is exactly the same as the one termed by Elliott *et al.* (equation 3 in reference [6]) as "general", except that here an explicit distinction is made between real and imaginary parts of the usually complex matrices. The above discussion indicates that in fact this controller objective represents a special case of the general robust multi-channel LMS control design problem. Again the weighting matrices are not chosen *a priori*, as is very often assumed, but they are the *a posteriori* result of solving the robust control design problem for a given robustness model. Choosing this robustness model, based on physical insight into the practical problem being considered, is easier than choosing the weighting matrices directly.

5. ROBUST STABILITY ANALYSIS OF THE ADAPTIVE LMS ALGORITHM

5.1. THE ROBUST STABILITY CONDITION

This section deals with the stability analysis of the adaptive multiple-channel LMS algorithm, whose behaviour is described (in a frequency domain implementation) by

$$\mathbf{u}(k+1) = (1 - \alpha\beta)\mathbf{u}(k) - \alpha\mathbf{G}_0^H\mathbf{e}(k), \tag{38}$$

where k denotes the discrete time, $\mathbf{G}_0$ is the model of the secondary path in the controller, α is the convergence coefficient, and β is the leak parameter [5, 6]. This adaptive algorithm is stable provided that [6]

$$\mathrm{Re}(\mathrm{eig}(\mathbf{G}_0^H\mathbf{G}_p(\boldsymbol{\Delta}) + \beta\mathbf{I})) > 0, \tag{39}$$

where $\mathbf{G}_p(\boldsymbol{\Delta})$ is the actual secondary path, which may be subject to unknown perturbations around its nominal state ($\mathbf{G}_0$) under practical operation conditions. The aim of this section is to derive a condition which guarantees that equation (39) is satisfied for all possible perturbations $\boldsymbol{\Delta}$ in the considered uncertainty model.

Consider now an autonomous linear system, described in the state space

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}. \tag{40}$$

This system is internally stable if all its poles lie on the left-hand side of the imaginary axis in the complex plane. This condition is equivalent to the following requirement on the eigenvalues of the state transition matrix $\mathbf{A}$

$$\mathrm{Re}(\mathrm{eig}(\mathbf{A})) < 0. \tag{41}$$

From linear system theory it is known that equation (41) is equivalent to the existence of a Lyapunov function, proving quadratic stability, or

$$\begin{aligned} &\exists \mathbf{P} \geqslant 0,\ \mathbf{Q} \geqslant 0, \\ &\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}. \end{aligned} \tag{42}$$

According to Boyd *et al.* [17], this Lyapunov stability condition can be written as an LMI feasibility problem in $\mathbf{P}$

$$\begin{aligned} &\text{find } \mathbf{P} \\ &\text{s.t. } \mathbf{P} \geqslant 0, \\ &\quad \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} < 0. \end{aligned} \tag{43}$$

The matrix $\mathbf{P}$ is the Hessian of the quadratic Lyapunov which is used to prove quadratic stability of system (40).

The similarity between conditions (39) and (41) allows the stability condition of the adaptive multiple-channel LMS algorithm to be posed as an uncertain LMI feasibility problem:

$$\begin{aligned} &\text{find } \mathbf{P} \\ &\text{s.t. } \mathbf{P} \geqslant 0 \\ &\quad (\mathbf{G}_0^{\mathrm{T}}\mathbf{G}_p(\boldsymbol{\Delta}) + \beta\mathbf{I})^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{G}_0^{\mathrm{T}}\mathbf{G}_p(\boldsymbol{\Delta}) + \beta\mathbf{I}) > 0, \end{aligned} \tag{44}$$

where $\mathbf{G}_0$ and $\mathbf{G}_p$ are the real-valued equivalents of $\mathbf{G}_0$ and $\mathbf{G}_p$.

This robust stability condition will be further elaborated in the following paragraphs for the unstructured and the structured perturbations case.

Note that the original stability condition (39) ignores the influence of the convergence coefficient α of the adaptive process. This effect could be taken into account by replacing equation (39) by the following stability condition [5, 6, 18] (for fixed α):

$$\mathrm{Re}(\mathrm{eig}(\mathbf{I} - \alpha(\mathbf{G}_0^{H}\mathbf{G}_p(\boldsymbol{\Delta}) + \beta\mathbf{I}))) > 0 \tag{45}$$

and adapting the subsequently derived conditions accordingly.

5.2. THE UNSTRUCTURED PERTURBATIONS CASE

The actual secondary path, $\mathbf{G}_p$, is assumed to belong to a bounded set around the secondary path model $\mathbf{G}_0$ on which the adaptive algorithm relies:

$$\mathbf{G}_p = \mathbf{G}_0 + \rho\boldsymbol{\Delta}, \quad \forall\boldsymbol{\Delta},\ \|\boldsymbol{\Delta}\| \leqslant 1. \tag{46}$$

Substituting equation (46) into equation (44) yields the following uncertain LMI problem:

$$\begin{aligned} &\text{find } \mathbf{P} \\ &\text{s.t. } \mathbf{P} \geqslant 0 \\ &\quad (\mathbf{G}_0^{\mathrm{T}}\mathbf{G}_0 + \beta\mathbf{I})^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{G}_0^{\mathrm{T}}\mathbf{G}_0 + \beta\mathbf{I}) + \rho\boldsymbol{\Delta}^{\mathrm{T}}\mathbf{G}_0\mathbf{P} + \rho\mathbf{P}\mathbf{G}_0^{\mathrm{T}}\boldsymbol{\Delta} > 0. \end{aligned} \tag{47}$$

Stability is assured only if condition (47) is satisfied for all possible perturbations $\boldsymbol{\Delta}$ whose norm is bounded by one. The following lemma, which is a corollary of the S-procedure and

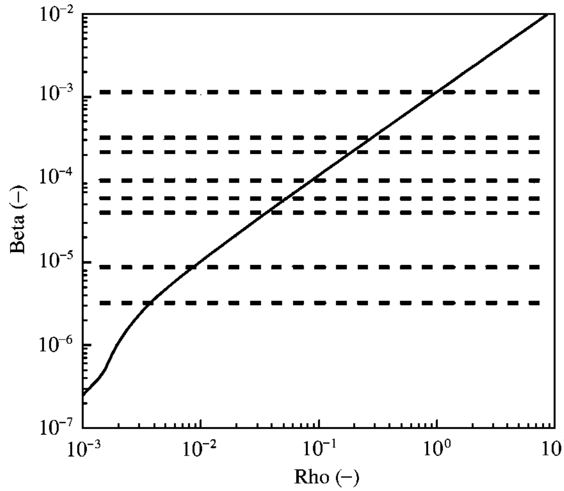

Figure 7. Minimal required leak β to guarantee the stability of an adaptive LMS algorithm when the secondary path is subject to unstructured perturbations of size ρ. The dashed grey lines represent the singular values of the secondary path.

whose proof is given by El Ghaoui and Lebret [16], allows the infinite number of conditions (39) to be replaced by one single equivalent condition.

Lemma 5.1. *Let* $\mathbf{M}_1 = \mathbf{M}_1^T, \mathbf{M}_2, \mathbf{M}_3$ *be real matrices of appropriate size. The inequality* $\mathbf{M}_1 + \mathbf{M}_2\mathbf{\Delta}\mathbf{M}_3 + \mathbf{M}_3^T\mathbf{\Delta}^T\mathbf{M}_2^T < 0$ *holds for every* $\mathbf{\Delta}$, $\|\mathbf{\Delta}\| \leqslant 1$, *if and only if* $\exists\lambda > 0$, $\lambda \in \Re$, *such that* $\mathbf{M}_1 + (1/\lambda)\mathbf{M}_2\mathbf{M}_2^T + \lambda\mathbf{M}_3\mathbf{M}_3^T < 0$.

Applying Lemma 5.1 to equation (47) yields an equivalent robust stability condition in the form of an LMI feasibility problem in $\mathbf{P}$ and λ:

$$\begin{aligned}&\exists\, \mathbf{P} \geqslant 0, \lambda > 0,\\ &[(\mathbf{G}_0^T\mathbf{G}_0 + \beta\mathbf{I})^T\mathbf{P} + \mathbf{P}(\mathbf{G}_0^T\mathbf{G}_0 + \beta\mathbf{I})] + \frac{1}{\lambda}\rho^2\mathbf{I} + \lambda\mathbf{P}\mathbf{G}_0^T\mathbf{G}_0\mathbf{P} > 0.\end{aligned} \tag{48}$$

By replacing $\lambda\mathbf{P}$ by $\mathbf{P}_0$, or alternatively by taking $\lambda = 1$, and by searching for the maximal ρ^2 for which equation (48) still holds, it is now possible to calculate the largest size of secondary path perturbations under which the stability of an adaptive LMS algorithm, characterized by a secondary path model $\mathbf{G}_0$ and a leak parameter β, is still guaranteed:

$$\begin{aligned}&\max_{\rho^2, \mathbf{P}_0} (\rho^2)\\ &\text{s.t. } \mathbf{P}_0 \geqslant 0\\ &[(\mathbf{G}_0^T\mathbf{G}_0 + \beta\mathbf{I})^T\mathbf{P}_0 + \mathbf{P}_0(\mathbf{G}_0^T\mathbf{G}_0 + \beta\mathbf{I})] + \mathbf{P}_0\mathbf{G}_0^T\mathbf{G}_0\mathbf{P}_0 + \rho^2\mathbf{I} > 0.\end{aligned} \tag{49}$$

The SDP problem (49) has been solved for 50 values of β ranging from 10^{-7} to 10^{-2}, and the corresponding stability limit ρ is plotted (on the abscissa) in Figure 7. Obviously, this figure shows that, in the range of the singular values of the secondary transfer path matrix, the leak parameter β should increase in proportion to the perturbation size and to the largest singular value of the secondary path.

Finally, note that this result can also be obtained by searching for the largest ρ (for a given $\mathbf{G}_0$ and β) for which the following algebraic Riccati equation has a solution:

$$(\mathbf{G}_0^{\mathrm{T}}\mathbf{G}_0 + \beta\mathbf{I})^{\mathrm{T}}\mathbf{P}_0 + \mathbf{P}_0(\mathbf{G}_0^{\mathrm{T}}\mathbf{G}_0 + \beta\mathbf{I}) + \mathbf{P}_0\mathbf{G}_0^{\mathrm{T}}\mathbf{G}_0\mathbf{P}_0 + \rho^2\mathbf{I} = 0. \tag{50}$$

5.3. THE STRUCTURED PERTURBATIONS CASE

The actual secondary path is now structured and defined as in section 2.2:

$$\begin{aligned} &\mathbf{G}_p = \mathbf{G}_0 + \mathbf{U}_0\boldsymbol{\Delta\Sigma}\mathbf{V}_0^H, \\ &\text{where } \boldsymbol{\Delta\Sigma} = \rho\boldsymbol{\Delta}\mathbf{W}, \\ &\text{and } \boldsymbol{\Delta} = \mathrm{diag}(\delta_1, \ldots, \delta_{N_e}), \text{ with } \delta_i \leqslant 1, \text{for } i = 1, \ldots, N_e. \end{aligned} \tag{51}$$

Substituting equation (51) into equation (44) then yields the following stability condition, to be satisfied for all diagonal $\boldsymbol{\Delta}$, with $\|\boldsymbol{\Delta}\| \leqslant 1$:

$$\begin{aligned} &\text{find } \mathbf{P} \\ &\text{s.t. } \mathbf{P} \geqslant 0 \\ &(\mathbf{G}_0^{\mathrm{T}}\mathbf{G}_0 + \beta\mathbf{I})^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{G}_0^{\mathrm{T}}\mathbf{G}_0 + \beta\mathbf{I}) + \rho\mathbf{V}\mathbf{W}^{\mathrm{T}}\boldsymbol{\Delta}^{\mathrm{T}}\mathbf{U}^{\mathrm{T}}\mathbf{G}_0\mathbf{P} + \rho\mathbf{P}\mathbf{G}_0^{\mathrm{T}}\mathbf{U}\boldsymbol{\Delta}\mathbf{W}\mathbf{V}^{\mathrm{T}} > 0 \end{aligned} \tag{52}$$

where U is the real-valued equivalent of $\mathbf{U}$.

The next lemma is a generalization of Lemma 5.1 for the structured perturbations case.

Lemma 5.2. *Let* $\mathbf{M}_1 = \mathbf{M}_1^{\mathrm{T}}, \mathbf{M}_2, \mathbf{M}_3$ *be real matrices of appropriate size. Let* $\boldsymbol{\Delta}_S$ *be a subspace of* $\Re^{N\times N}$ *with the same structure as the uncertainty block* $\boldsymbol{\Delta}$. *The inequality* $\mathbf{M}_1 + \mathbf{M}_2\boldsymbol{\Delta}\mathbf{M}_3 + \mathbf{M}_3^{\mathrm{T}}\boldsymbol{\Delta}^{\mathrm{T}}\mathbf{M}_2^{\mathrm{T}} < 0$ *holds for every* $\boldsymbol{\Delta} \in \boldsymbol{\Delta}_S$, *with* $\|\boldsymbol{\Delta}\| \leqslant 1$, *if* $\exists \mathbf{S} \in \mathsf{S}, \mathbf{T} \in \mathsf{T}$, *defined as in equation* (30), *such that*

$$\mathbf{S} > 0$$

$$\begin{bmatrix} \mathbf{M}_1 - \mathbf{M}_2\mathbf{S}\mathbf{M}_2^{\mathrm{T}} & \mathbf{M}_3^{\mathrm{T}} + \mathbf{M}_2\mathbf{T} \\ \mathbf{M}_3 - \mathbf{T}\mathbf{M}_2^{\mathrm{T}} & \mathbf{S} \end{bmatrix} > 0.$$

Applying this lemma, whose proof is detailed by El Ghaoui and Lebret [16], to equation (52) yields only a sufficient condition (and no longer necessary as in the unstructured perturbations case) for the robust stability of the adaptive LMS algorithm. Due to the diagonality of the perturbation block, the subspace S only contains diagonal matrices, and the subspace T only the null matrix. The stability condition then becomes an LMI feasibility problem in $\mathbf{S}$ and $\mathbf{P}$:

$$\begin{aligned} &\text{find } \mathbf{S} \text{ diagonal}, \mathbf{P} \\ &\text{s.t.} \quad \mathbf{P} \geqslant 0, \mathbf{S} > 0, \\ &\begin{bmatrix} (\mathbf{G}_0^{\mathrm{T}}\mathbf{G}_0 + \beta\mathbf{I})^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{G}_0^{\mathrm{T}}\mathbf{G}_0 + \beta\mathbf{I}) - \rho^2\mathbf{G}_0^{\mathrm{T}}\mathbf{U}\mathbf{S}\mathbf{U}^{\mathrm{T}}\mathbf{G}_0 & \mathbf{P}\mathbf{V}\mathbf{W}^{\mathrm{T}} \\ \mathbf{W}\mathbf{V}^{\mathrm{T}}\mathbf{P} & \mathbf{S} \end{bmatrix} > 0. \end{aligned} \tag{53}$$

Note that only the relative magnitude of $\mathbf{S}$ and $\mathbf{P}$ in equation (53) are relevant, not their absolute magnitude. Therefore, the feasibility problem (53) can be replaced by an equivalent maximization problem with improved numerical properties:

$$\max_{\gamma,\mathbf{P},\mathbf{S}} \gamma$$

$$\text{s.t.}\quad \mathbf{P}-\gamma\mathbf{I}\geqslant 0,\ \mathbf{I}-\mathbf{P}\geqslant 0,\ \mathbf{S}>0, \mathbf{S}\text{ diagonal} \tag{54}$$

$$\begin{bmatrix} (\mathbf{G}_0^{\mathrm{T}}\mathbf{G}_0+\beta\mathbf{I})^{\mathrm{T}}\mathbf{P}+\mathbf{P}(\mathbf{G}_0^{\mathrm{T}}\mathbf{G}_0+\beta\mathbf{I})-\rho^2\mathbf{G}_0^{\mathrm{T}}\mathbf{U}\mathbf{S}\mathbf{U}^{\mathrm{T}}\mathbf{G}_0 & \mathbf{P}\mathbf{V}\mathbf{W}^{\mathrm{T}} \\ \mathbf{W}\mathbf{V}^{\mathrm{T}}\mathbf{P} & \mathbf{S} \end{bmatrix} > 0.$$

Stability is thus assured when the objective γ in equation (54) is positive.

Given an exact model $\mathbf{G}_0$ of the nominal secondary path, and given a set of all actually possible secondary paths, described by a perturbation structure $\mathbf{W}$ and size ρ around the nominal secondary path, it is now possible to determine the smallest leak parameter β_s which guarantees stability of the adaptive algorithm under all actually occurring operation conditions by solving the SDP problem (54) for subsequent β's and choosing the smallest one for which γ is positive. A similar problem is discussed by Omoto and Elliott [8]. They first present a conservative stability condition which they derive from the Gershgorin circle theorem [9]. When leak is present, this condition can be written as

$$\sigma_c+\frac{\beta}{\sigma_c} > \sum_{e=1}^{N_c}|\boldsymbol{\Delta\Sigma}_{e,c}| \quad \text{for } c=1,\ldots,N_c, \tag{55}$$

where σ_c is the cth singular value of the secondary path matrix $\mathbf{G}_0$. Taking into account the structure and the size of the perturbation in equation (51), the right-hand side of (55) is bounded by

$$\rho\sum_{e=1}^{N_c}\mathbf{W}_{e,c} > \sum_{e=1}^{N_c}|\boldsymbol{\Delta\Sigma}_{e,c}| \quad \text{for } c=1,\ldots,N_c. \tag{56}$$

Combining equations (55) and (56) allows the lower bound on the leak parameter to be calculated to satisfy the stability condition derived from the Gershgorin circle theorem

$$\beta_g=\max_{c=1,\ldots,N_c}\left(\sigma_c\rho\sum_{e=1}^{N_c}\mathbf{W}_{e,c}-\sigma_c^2\right). \tag{57}$$

As an alternative to this conservative bound, Omoto and Elliott [8] propose to consider only the diagonal terms in $\boldsymbol{\Delta\Sigma}$, assuming that they are more important for determining stability of the adaptive algorithm:

$$\sigma_c+\frac{\beta}{\sigma_c} > -\operatorname{Re}(\boldsymbol{\Delta\Sigma}_{c,c}) \quad \text{for } c=1,\ldots,N_c. \tag{58}$$

Taking into account the structure and the size of the perturbation in equation (51), condition (58) yields the following lower bound on the leak parameter:

$$\beta_d=\max_{c=1,\ldots,N_c}(\sigma_c\rho\mathbf{W}_{e,c}-\sigma_c^2). \tag{59}$$

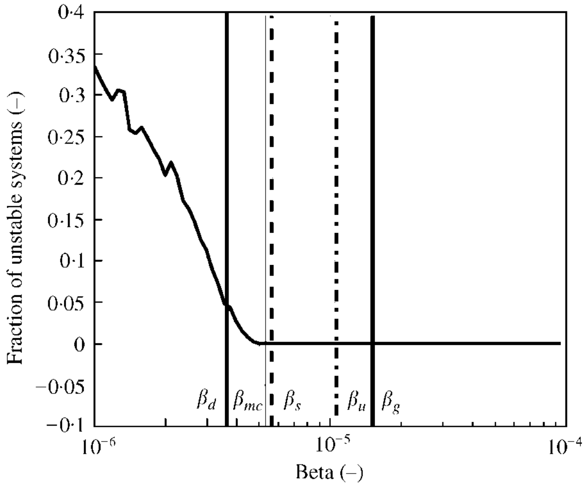

Figure 8. Fraction of unstable adaptive algorithms in 2000 simulations with structured perturbations on the secondary path ($\rho = 10^{-2}$). The vertical lines indicate different lower bounds on the leak to guarantee stability (diagonal of $\mathbf{\Delta\Sigma}$: β_d, from the simulations: β_{mc}, theoretical limit for structured perturbations: β_s, theoretical limit for unstructured perturbations: β_u, Gershgorin circle theorem: β_g).

These different theoretical lower bounds on the leak parameter have been confronted with a Monte Carlo simulation. The basic stability condition (39) is checked for 2000 different secondary paths, $\mathbf{G}_p$, with structured perturbations as described in equation (51) with a perturbation size ρ equal to 10^{-2}, at each single-leak parameter, and the fraction of unstable systems is plotted in Figure 8 for 80 logarithmically spaced values of β between 10^{-6} and 10^{-4}. The lowest leak for which none of the perturbed systems resulted in an unstable algorithm is represented in Figure 8 by the thin vertical line with index β_{mc}. The theoretical lower bounds introduced in this section are also represented by different vertical lines, as well as the theoretical bound β_u calculated using the method from the previous section, thus ignoring any structure in the perturbation. This latter bound β_u can be read directly from Figure 7 (at $\rho = 10^{-2}$). The results in Figure 8 clearly indicate that the theoretical lower bound, β_s, computed by taking into account structure in the perturbation with the method presented here, is larger than, but very close to the one observed from the Monte Carlo simulations. Note that, from a theoretical point of view, this method yields sufficient stability conditions, and thus only produces a guaranteed upper limit (on the lower bound) on the necessary leak. In simulations, this upper limit proves to be very tight. The method from the previous section, which yields sufficient and necessary stability conditions for unstructured perturbations, produces an over-conservative lower bound on the leak when the perturbations are actually structured. The lower bound, obtained from the Gershgorin circle theorem is even more conservative, though it takes into account the perturbation structure. The stability criterion which only takes into account the diagonal elements of $\mathbf{\Delta\Sigma}$, yields an over-optimistic lower bound on the required leak. The new method presented here allows an adaptive multiple-channel leaky LMS algorithm be designed which is guaranteed to be stable in the face of structured secondary path perturbations, but which does not reduce the performance too much by using too high a leak parameter.

6. CONCLUSIONS

The problem of designing robust feedforward control systems for active noise control applications with single-tone disturbances is considered in this paper. The main concept in the paper is the formulation of the nominal performance or the nominal stability analysis of the LMS control algorithm as a semidefinite programming (SDP) problem, followed by the application of the S-procedure to derive new SDP formulations for these problems when the data matrices are subject to uncertainty. These robust SDP formulations yield exact robust solutions in the unstructured perturbations case, and upper bounds on the exact robust solutions in the structured perturbations case.

An SDP problem is a convex optimization problem, consisting of a linear objective function subject to linear matrix inequality (LMI) constraints. Such SDP formulations are attractive from a practical point of view, because recently developed interior-point methods solve SDP problems with a few hundred variables in affordable computation times (seconds up to a few minutes).

Firstly, an SDP formulation is presented for the design of multi-channel LMS algorithms with limited-capacity secondary sources. Simulations show that this SDP formulation is an order of magnitude more computationally efficient than the usual non-linear constrained optimization formulations.

Secondly, the design of robust LMS algorithms is presented as an SDP problem. These algorithms minimize the worst-case control error in the presence of unknown but norm-bounded perturbations on the secondary path model and on the primary field. The resulting controllers are exact solutions to the robust control design problem, except in the most general case of structured perturbations when they only minimize an upper bound on the worst-case residual control error. Monte Carlo simulations confirm the theoretical results.

Thirdly, SDP formulations are proposed to compute guaranteed stability limits for the adaptive multiple-channel leaky LMS algorithm in the presence of both unstructured and structured perturbations on the secondary path. Monte Carlo simulations show that the obtained stability limits are much more reliable than previously used limits, based, for example, on the Gershgorin circle theorem. The method presented here allows an adaptive multiple-channel leaky LMS algorithm to be designed which is guaranteed to be stable in the face of structured secondary path perturbations, but which does not reduce the performance too much by using an excessive leak parameter.

The main merit of the paper is the combination of different existing theoretical results into a framework which can be usefully applied for designing robust active noise control systems. Nevertheless, additional research work is still required, in particular to define sensible estimates of possible perturbation structures in real-world applications.

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APPENDIX A: THE SCHUR COMPLEMENT

Consider the partitioned symmetric matrix $\mathbf{X}$:

$$\mathbf{X} = \mathbf{X}^{\mathrm{T}} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^{\mathrm{T}} & \mathbf{C} \end{bmatrix}. \tag{A1}$$

Provided that $\mathbf{A} \neq 0$, the Schur complement of $\mathbf{A}$ in $\mathbf{X}$ is given by

$$\mathbf{S} = \mathbf{C} - \mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{B}. \tag{A2}$$

The following Schur complement lemma [17] is frequently used to convert a non-linear (convex) inequality into an LMI. In the formulation of the lemma, '>0' indicates that the matrix is symmetric and positive definite, and '$\geqslant 0$' indicates that the matrix is symmetric and positive semidefinite.

Schur Complement Lemma (*without proof*)

- $\mathbf{X} > 0$ *if and only if* $\mathbf{A} > 0$ *and* $\mathbf{S} > 0$.
- *If* $\mathbf{A} > 0$, *then* $\mathbf{X} \geqslant 0$ *if and only if* $\mathbf{S} \geqslant 0$.